The solutions below are in response to student requests.

HW5

2. Let V be a finite dimensional vector space and $T, U : V \to V$ be non-zero linear maps that satisfy $R(T) \cap R(U) = \{0_V\}$. Prove that T and U are linearly independent in $\mathcal{L}(V)$, the space of linear maps from V to V.

pf. Suppose that $T = \lambda U$ for some $\lambda \in \mathbb{F}$. Let $x \in V$, $x \neq 0$. Set $T(x) = \lambda U(x) = U(\lambda x) = w$. Then $w \in R(T) \cap U(T)$, so $w = 0$. Since x was arbitrarily chosen, $T(x) = 0$ for all x. But T was assumed to be a non-zero map, so $T \neq \lambda U$.

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4. Let B be a fixed $n \times n$ matrix with entries in F, and define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = BAB^{-1}$.

HW6

(a) Show that Φ is linear (Hint: use Theorem 2.10(a) from the book).
(b) Show that Φ is an isomorphism.

5. Suppose V, W are finite-dimensional vector spaces and $T : V \to W$ is an isomorphism. Suppose V_0 is a subspace of V. Show that $T(V_0)$ (that is, the set of all vectors of the form $T(v)$ for $v \in V_0$) is a subspace of W of the same dimension as V_0.

4 (a)
$$\begin{aligned}\Phi(\lambda A_1 + A_2) &= B(\lambda A_1 + A_2)B^{-1} \\ &= B\lambda A_1 B^{-1} + BA_2B^{-1} \\ &= \lambda BA_1B^{-1} + BA_2B^{-1} \\ &= \lambda\Phi(A_1) + \Phi(A_2).\end{aligned}$$

(b) Suppose $A \in M_2(F)$ and $\Phi(A) = 0$. Then $BAB^{-1} = 0$, so $A = B^{-1}0B = 0$. Hence Φ is one-to-one.
Let $C \in M_2(\mathbb{F})$. Then $B^{-1}CB \in M_2(\mathbb{F})$.
$\Phi(B^{-1}CB) = B(B^{-1}CB)B^{-1} = I_nCI_n = C$. Hence Φ is onto.

(You could also show that $\hat{\Phi} : M_2(\mathbb{F})$, $\hat{\Phi}(A) = B^{-1}AB$, is an inverse for Φ.)

5 Since $0_V \in V_0$, $T(0_V) = 0_W \in T(V_0)$. Suppose $w_1, w_2 \in T(V_0)$. Let $w_1 = T(v_1)$ and $w_2 = T(v_2)$ where $v_1, v_2 \in V_0$. Then $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$. Since V_0 is a subspace, $v_1 + v_2 \in V_0$. So $w_1 + w_2 \in T(V_0)$.
Let $\lambda \in \mathbb{F}$. Then $\lambda w_1 = \lambda T(v_1) = T(\lambda v_1)$. Since V_0 is a subspace, $\lambda v_1 \in V_0$. So $\lambda w_1 \in T(V_0)$. As $\vec{0} \in T(V_0)$ and $T(V_0)$ is closed under vector addition and scalar multiplication, $T(V_0)$ is a subspace of W.

Let $\beta = \{v_1, v_2, \ldots, v_k\}$ be a basis for V_0. Then $T(\beta)$ spans $T(V_0)$, as $T(V_0) = R(T|_{V_0})$, where $T|_{V_0}$ denotes T restricted to V_0.
Suppose $a_1T(v_1) + \cdots + a_kT(v_k) = 0$. Then $T(a_1v_1 + \cdots + a_kv_k) = 0$, so $a_1v_1 + \cdots + a_kv_k \in \operatorname{Ker}(T)$. But T is one-to-one, so $a_1v_1 + \cdots + a_kv_k = 0$. Since β is independent, $a_1 = a_2 = \cdots = a_k = 0$. So $T(\beta)$ is independent. Hence $T(\beta)$ is a basis for $T(V_0)$, and V_0 and $T(V_0)$ have equal dimension. (We also see that one-to-one transformations carry independent sets to independent sets.)

HW 7

1. Let U and W be vector spaces. We define the product $U \times W$ to mean the set of ordered pairs (u, w) with $u \in U$ and $w \in W$ with operations $(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2)$ and $\lambda(u, w) = (\lambda u, \lambda w)$. It is easy to see that $U \times W$ is a vector space under these operations.

   (a) Show that $\dim(U \times W) = \dim U + \dim W$.

   (b) Now suppose that U and W are both subspaces of a vector space V and let $T : U \times W \longrightarrow V$ be the map sending (u, w) to $u + w$. Show that $\dim N(T) = \dim(U \cap W)$.

(a) Suppose one of U, W is not finite-dimensional. Then $U \times W$ is also infinite dimensional.
Suppose U, W both finite dimensional. Let $\beta_u = \{u_1, \dots u_k\}$ be a basis for U and $\beta_W = \{w_1, \dots, w_m\}$ a basis for W. We claim that $\beta = \{(u_1, 0_w), (u_2, 0_w), \dots, (u_k, 0_w), (0_v, w_1), \dots, (0_v, w_m)\}$ is a basis for $U \times W$. We can see that β spans $U \times W$, as if $(u, w) \in U \times W$, $u = \sum_{i=1}^{k} a_i u_i$, $w = \sum_{j=1}^{m} b_j w_j$, then (u, w)
$= (\sum a_i u_i, \sum b_j w_j)$
$= \sum a_i (u_i, 0_w) + \sum b_j (0, w_j)$.
Suppose $\lambda_1 (u_1, 0_w) + \dots + \lambda_k (u_k, 0_w) + \alpha_1 (0_v, w_1) + \dots + \alpha_m (0_v, w_m) = 0$.
Then $\lambda_1 u_1 + \dots + \lambda_k u_k = 0$ and $\alpha_1 w_1 + \dots + \alpha_m w_m = 0$. Hence $\lambda_1 = \dots = \lambda_k = \alpha_1 = \dots = \alpha_m = 0$, so β is independent.
Then $\dim(U \times W) = k + m = \dim U + \dim W$.

(b) $N(T) = \{(u, w) \in U \times W \mid w = -u\}$. If $-u \in W$, then $u \in W$, so $u, w \in U \cap W$. Then $N(T) = \{(u, -u) \mid u \in U \cap W\}$. To see this,
Let $\Phi : U \cap W \longrightarrow N(T)$ be given by $\Phi(u) = (u, -u)$. Then Φ is one-to-one, as if $\Phi(u) = 0$, then $(u, -u) = (0, 0)$, so $u = 0$.
Φ is onto, as if $(u, w) \in N(T)$, then $w = -u$ and $u \in U \cap W$.
So Φ is an isomorphism, meaning that $U \cap W$, $N(T)$ have equal dimension.

HW 8

2. (a) Let A be an $m \times n$ (m rows, n columns) matrix and B be an $n \times p$ matrix. Suppose that $\text{rank}(A) = m$ and $\text{rank}(B) = n$. Find the rank of AB. Justify your answer.

   (b) Prove or give a counter example to the following statement: If the $m \times n$ (m rows, n columns) matrix A has rank m, then the system $Ax = b$ is consistent for any choice of b.

   (c) Prove or give a counter example to the following statement: For two $m \times n$ matrices A, B, we must have $\text{rank}(A + B) \le \min\{\text{rank}(A), \text{rank}(B)\}$.

(a) $\text{rank}(AB) = \text{rank}(L_{AB}) = \text{rank}(L_A L_B)$. Since $L_A : \mathbb{F}^n \to \mathbb{F}^m$ has rank m, L_A is onto. Similarly L_B is onto. Hence L_{AB} is the composition of onto maps, and must be onto. $L_{AB} : \mathbb{F}^n \to \mathbb{F}^m$, so $\text{rank}(L_{AB}) = m = \text{rank}(AB)$.
(See HW5 Q3).

(b) This is true. Since m is the maximum rank $(A|b)$, and A has rank m, $(A|b)$ also has rank m.

(c) This is false. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.